# SYMMETRIC CLOSURE IN MODULES AND RINGS

ANDRÉ G. LEROY* AND MEHRDAD NASERNEJAD

*Univ. Artois, UR 2462, Laboratoire de Mathématique de Lens (LML), F-62300 Lens, France*

Abstract. We introduce both for modules and rings classes of elements that are strongly connected to commutativity classes as defined in [1] [M. Abdi and A. G. Leroy, *Graphs of commutatively closed sets*, Linear Multilinear Algebra, 2021] and [2] [D. Alghazzawi and A. G. Leroy, *Commutatively closed sets in rings*, Comm. Algebra **47** (2019), no. 4, 1629–1641]. We define a graph structure on the classes leading to a notion of distance for elements of a class. Examples are presented all along the paper, showing the interest of the notions.

## 1. Introduction and notations

This paper is essentially concerned with notions around factorizations in general noncommutative rings. The starting points for our considerations were the papers [1] and [2], the definition of symmetric rings that first appears in J. Lambek [11], and the usual UFD notions (cf. [5]).

While considering uniqueness of the factorization of an element $a$ in a ring $R$, it is natural to attach to $a$ all the elements of $R$ that can be obtained by permuting the factors in factorizations of $a$. It is thus natural to consider the set

$$\{b \in R \mid \exists n \in \mathbb{N}, \exists (a_1, \ldots, a_n) \in R^n, \text{ with } a = a_1 \cdots a_n \text{ and } b = a_{\sigma(1)} \cdots a_{\sigma(n)}\},$$

where $\sigma \in S_n$. This set will be denoted $\widehat{\{a\}}_1$. If $b \in \widehat{\{a\}}_1$, we will write $b \widehat{\sim}_1 a$. Continuing this process we may attach to an element $a \in R$ its symmetric closure $\widehat{\{a\}}$ defined by

$$\widehat{\{a\}} = \{x \in R \mid \exists l \in \mathbb{N}, b_1, \ldots, b_l \in R \text{ with } a \widehat{\sim}_1 b_1 \widehat{\sim}_1 b_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 b_l = x\}.$$

Of course, when the ring $R$ is commutative, we have $\widehat{\{a\}} = \{a\}$. We will see that the symmetric closure admits another construction. This construction is strongly connected to the notion of symmetric rings that was introduced by Lambek ([11]). In general, $\widehat{\{a\}}$ measures the "local" noncommutativity of $R$. The two ways of constructing the symmetric closure give two distances inside the symmetric closure of an element offering two ways of measuring this "local" noncommutativity of the

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*Corresponding author.

E-mail addresses: andre.leroy@univ-artois.fr and m_nasernejad@yahoo.com.

ring $R$. This naturally leads to the introduction of the diameter of the garphs. Let us now briefly describe the content of the sections.

We start with definitions based on modules, in Section 2. Factorizations of an element $m$ in a module $M_R$ are based on the notion of divisibility of $m$ by elements of $R$ (cf. [10], P. 70). The closure $\widehat{\{m\}}$ of an element $m \in M_R$ is defined (cf. Definitions 2.1). The derived group $U(R)'$ of the unit group of the base ring $R$ appears to be an essential tool. Special attention is devoted to the case of cyclic modules and to the class of atoms of the module.

In Section 3, we consider two different factorization chains attached to an element of a ring $R$. The closure notions corresponding to these factorization chains are in fact the same. This closure operation leads to a topology on $R$, and hence we obtain two ways to analyze this topology. We give a description of these classes in different rings. We also remark some relations with the commutative closure introduced in [2]. Among others, we compute the class of the identity for Dedekind-finite rings and show that $\widehat{\{0\}}$ is equal to the derived group $U(R)'$ of the unit group $U(R)$ of the ring $R$. Using the Dieudonné determinant we describe all the classes of the ring of matrices over a division ring and, more generally, over an Artinian semisimple ring.

In the last section of the paper, we define a graph structure on each symmetry class. Two notions of distances are defined in these graphs. These distances and the diameters of the graphs are compared. The case of matrices over division rings is studied. It offers connections between our distances and subject such as the number of multiplicative commutators needed to express an element of $SL_n(D)$, the number of idempotents that appear in the expression of a singular matrix as product of idempotent matrices and the number of conjugates necessary to express a singular matrix in terms of conjugates of another singular matrix of higher rank. Once again this part is largely influenced by the results obtained in [1] and [2].

Throughout this paper, the symbol $\mathbb{N}$ stands for the positive integers. In addition, we assume that all rings are unitary. For a ring $R$, $U(R)$ and $U(R)'$ will denote the group of units of $R$ and its derived group, respectively. The set of nilpotent elements of a ring $R$ will be denoted by $N(R)$. In addition, the notation $\lceil x \rceil$ denotes the *ceiling function* of a real number $x$, which is the least integer greater than or equal to $x$. If $D$ is a division ring and $n \in \mathbb{N}$, the Dieudonné determinant of a matrix $A \in M_n(D)$ will be denoted as $\mathrm{Det}(A)$.

## 2. Symmetric closure of modules

We will decompose an $R$-module into subsets that correspond to equivalence classes of an equivalence relation and are strongly related to the factorization in the ring $R$ and the structure of the module. This will lead to a distance between elements in these classes. The main definition that follows is inspired by the notion of symmetric rings.

**Definitions 2.1.** *Let $M$ be a right $R$-module. Two elements $m, n \in M$ are symmetrically connected if there exist $m' \in M$ and $a, b \in R$ such that $m = m'ab$ and $n = m'ba$. We denote this situation by $m \overset{1}{\sim} n$.*

*Two elements $m, n \in M$ are symmetrically related if there exists a finite chain of symmetrically connected elements $m = m_0 \overset{1}{\sim} m_1 \overset{1}{\sim} \dots \overset{1}{\sim} m_l = n$. We will write $m \sim n$ when $m$ and $n$ are symmetrically related.*

*For an element $m \in M$, we put $\widehat{\{m\}} := \{n \in M \mid n \sim m\}$, and we say that an element $m \in M$ is symmetrically closed when $\widehat{\{m\}} = \{m\}$.*

We first give some concrete examples.

**Examples 2.2.** (1) If $R$ is a commutative ring, then two elements of a module $M_R$ are symmetrically connected if and only if they are equal.
(2) If $M = R_R$, then $\widehat{\{0\}} = \{0\}$ if and only if the ring $R$ is symmetric. These kind of rings were introduced by J. Lambek in [11].
(3) If $D$ is a noncommutative division ring and two nonzero elements $x, y \in D$ are symmetrically connected, then $x = x'uv$ and $y = x'vu$ so that we get $y = x'uvv^{-1}u^{-1}vu = x[v^{-1}, u^{-1}]$, where $[u^{-1}, v^{-1}] = u^{-1}v^{-1}uv$ is a multiplicative commutator. We will generalize this example later in the next proposition.
(4) If $m \in M_R$, then we always have $m(U(R))' \subseteq \widehat{\{m\}}$; indeed, if $u, v \in U(R)$, then $m[u, v] = muvu^{-1}v^{-1} \overset{1}{\sim} muvv^{-1}u^{-1} = m$. This easily yields the desired inclusion.

In order to determine the closure of an element $m$ in a module $M$, we need to write $m$ as a product $m = m'r$ for some $r \in R$. Notice that in this case we obtain that $\mathrm{rann}(r) := \{s \in R \mid rs = 0\} \subseteq \mathrm{ann}(m) = \{x \in R \mid mx = 0\}$. This leads to the following definition:

**Definition 2.3.** *Let $m$ be a nonzero element in a module $M_R$. We say that $r$ divides $m$ (or that $m$ is divisible by $r$) if* $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$ *and there exists $m' \in M$ such that $m = m'r$.*

*A module $M_R$ is divisible if for any $m \in M$ and any $r \in R$ such that* $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$, *$r$ divides $m$.*

*We will say that an element $m \in M$ is an atom if the only $r \in R$ that divides $m$ are the invertible elements of $R$, i.e., the elements of $U(R)$.*

For an element $m \in M_R$, we introduce the following notations:

$$\mathrm{Div}_M(m) = \{m' \in M \mid m \in m'R\},$$

and

$$\mathrm{Div}_R(m) = \{r \in R \mid \exists m' \in \mathrm{Div}_M(m) : m = m'r\}.$$

Let us recall that for an element $a \in R$, we denote by $\overline{\{a\}}$ the commutative closure of $a$. To define it, we first consider $\{a\}_1 = \{xy \mid yx = a\}$. We then define, for any $i \geq 1$, $\{a\}_{i+1} = \{xy \mid yx \in \{a\}_i\}$, and finally $\overline{\{a\}} = \bigcup_{l \geq 1} \{a\}_l$. This notion has been studied in [1] and [2].

**Proposition 2.4.** (i) *The relation $\sim$ on a module $M$ is an equivalence relation on $M$.*
(ii) *Let $V_D$ be a vector space over a noncommutative division ring $D$. Two vectors $v, w \in V$ are symmetrically related if and only if $w \in v(D^*)'$, where $(D^*)'$ is the derived group of the multiplicative group $D \setminus \{0\}$.*
(iii) *If $M = mR$ is a cyclic module, then*

$$\mathrm{Div}_R(m) = \{r \in R \mid \exists s \in R : 1 - sr \in \mathrm{ann}_R(m)\}.$$

(iv) *If $r \in \mathrm{Div}_R(m)$ and $n \in \mathrm{Div}_M(m)$ are such that $m = nr$, then we have $n + \mathrm{ann}_M(r) \subseteq \mathrm{Div}_M(m)$, where $\mathrm{ann}_M(r) = \{p \in M \mid pr = 0\}$.*

(v) *If an element* $m \in M$ *is divisible by* $r \in R$*, then* $m = nr$ *and* $n\widehat{\{r\}} \subseteq \widehat{\{m\}}$*.*

(vi) *If* $m, n \in M_R$ *be such that there exists an element* $x \in U(R)'$ *such that* $m = nx$*, then* $m \sim n$*. So that we have* $m(U(R))' \subseteq \widehat{\{m\}}$*.*

*Proof.* (i) Reflexivity is due to the fact that $1 \in R$. The symmetry of $\overset{1}{\sim}$ is straightforward, and the symmetry of $\sim$ follows. The transitive closure of the relation $\overset{1}{\sim}$ is exactly the relation $\sim$.

(ii) Suppose that $v = v'st$ and $w = v'ts$ for some $v' \in V$ and $s, t \in D \setminus \{0\}$. Then $w = v'ts = v'st[t^{-1}, s^{-1}] = v[t^{-1}, s^{-1}]$, where $[t^{-1}, s^{-1}] = t^{-1}s^{-1}ts$ is the multiplicative commutator. This leads quickly to the fact that two vectors $v, w \in V$ are symmetrically related if and only if $w \in v(D^*)'$, where $(D^*)'$ stands for the derived group of the multiplicative group of $D^*$.

(iii) Let $r \in \mathrm{Div}_R(m)$ and $m' \in M$ be such that $m = m'r$. Since $M = mR$, there exists $s \in R$ such that $m' = ms$, and we have $m = m'r = msr$ so that $1 - sr \in \mathrm{ann}_R(m)$. Conversely, if $1 - sr \in \mathrm{ann}_R(m)$, then $m = msr = m'r$ for $m' = ms$, and hence $r \in \mathrm{Div}_R(m)$.

(iv) We have $m = nr$ and, for any $m' \in \mathrm{ann}_M(r)$, we also have $(n + m')r = nr + m'r = nr = m$ so that $n + m' \in \mathrm{Div}_M(m)$.

(v) Let $s \in \widehat{\{r\}}$. We want to show that $ns \in \widehat{\{m\}}$. Clearly, it is enough to do this for elements $s$ such that $s \overset{1}{\sim} r$. So, let us suppose that there exist $x, y, z \in R$ such that $r = xyz$ and $s = xzy$. Then we have $m = nr = nxyz$ and $ns = nxzy \overset{1}{\sim} nxyz = m$, showing that $ns \in \widehat{\{m\}}$.

(vi) We first remark that if there exist $u, v \in U(R)$ such that $m = n[u, v]$, then $m = nuvu^{-1}v^{-1} \overset{1}{\sim} nuvv^{-1}u^{-1} = n$. Hence, if two elements of $M_R$ differs by a commutator, then they are symmetrically related. This yields the proof since, by definition, the derived group is generated by multiplicative commutators. □

As a consequence of statement (v) above, we get the following corollary.

**Corollary 2.5.** *Let* $m \in M_R$*. Then we have*

$$\bigcup_{(m',r)\in M\times R,\ where\ m=m'r} m'\widehat{\{r\}} \subseteq \widehat{\{m\}}.$$

Another interesting corollary is based on statement (vi) in Proposition 2.4.

**Corollary 2.6.** *If there exists a module* $M$ *and a torsion-free element* $m \in M$ *such that* $\widehat{\{m\}} = \{m\}$*, then the invertible elements of* $R$ *are central.*

*Proof.* Thanks to Proposition 2.4(vi), we have $mU(R)' \subseteq \widehat{\{m\}} = \{m\}$, and hence $U(R)' = \{1\}$, as required. □

**Examples 2.7.** (1) Let $F$ be a field and consider, for $n \geq 1$, the row space $M = F^n$ as a right module over the ring of matrices $M_n(F)$. For any two nonzero vectors $u, v \in F^n$ there always exists a matrix $E \in SL_n(F)$ such that $u = Ev$. Since $E$ is the derived group of $GL_n(F)$ (except when $F = \mathbb{F}_2$ and $n = 2$), we deduce that any two nonzero vectors from $F^n$ are always in the same class, i.e., $u \sim v$ for any $u, v \in F^n \setminus \{0\}$.

(2) The equality in the previous corollary is not true as it can be seen while computing the symmetric class of zero in the ring $R = \frac{k\langle X,Y\rangle}{(XY)}$, where $k$ is a field (note that the invertible elements of $R$ are the nonzero elements of $k$).

**Remarks 2.8.** *(1) If $r \in R$ divides $m \in M$, say $m = m'r$ for some $m' \in M$, then the elements of $M$ that are divible by $r$ are exactly those of the form $m' + \mathrm{ann}_M(r)$. This set will be written as $[\frac{m}{r}]$.*

*(2) If $m \in M$ and $x \in \mathrm{ann}(m)$, then $m(1-x) = m$, and we conclude that $ms \in \widehat{\{m\}}$ for any $s \overset{1}{\sim} 1-x$.*

Both the structure of $M$ as a module and the structure of the ring $R$ have an impact on the classes. For the influence of the module, let us examine the case of cyclic modules.

**Proposition 2.9.** *Let $M = mR$ be a cyclic module. Then for any $r, s \in R$, we have $mr \overset{1}{\sim} ms$ if and only if there exist $x, y \in R$ such that $x \overset{1}{\sim} y$ and $m(r-x) = m(s-y) = 0$.*

*Proof.* Suppose that $mr \overset{1}{\sim} ms$. Then there exist $m' \in M$ and $p, q \in R$ such that $mr = m'pq$ and $ms = m'qp$. Since $M = mR$, there exists $l \in R$ such that $m' = ml$. Putting $x := lpq$ and $y := lqp$, we get $x \overset{1}{\sim} y$, $mr = mlpq = mx$, and $ms = mlqp = my$. Retracing our steps, we obtain the converse statement. □

The structure of the ring $R$ is also important while computing the symmetric classes of an element from a module. Indeed, we have already remaked that the derived group $u(R)'$ of the group of invertible elements of $R$ was crucial (cf. Proposition 2.4). To show further this influence, we consider the case of a modules $M_R$ over a (von Neumann) regular ring $R$. Recall that an element $r$ in a ring $R$ is called *regular* if there exists $x$ in $R$ such that $r = rxr$. We denote the set of regular elements by $Reg(R)$. The ring $R$ is *(von Neumann) regular* if $Reg(R) = R$.

**Lemma 2.10.** *Let $m \in M_R$ be a nonzero element in an $R$-module $M_R$ and $r \in Reg(R)$ such that $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$. Then $r$ divides $m$.*

*Proof.* Since $r \in Reg(R)$, there exists $x \in R$ such that $r = rxr$, and hence $1 - xr \in \mathrm{rann}(r) \subseteq \mathrm{ann}(m)$. This gives $m = mxr$, and so $m = m'r$ for $m' = mx$. This completes the proof. □

This immediately leads to the following corollary.

**Corollary 2.11.** *Let $m \in M_R$ be a nonzero element of a right $R$-module. Suppose $r \in Reg(R)$ is such that $\mathrm{rann}(r) \subseteq \mathrm{ann}(m)$. Then $mx\widehat{\{r\}} \subseteq \widehat{\{m\}}$, where $x$ is any quasi inverse of $r$, i.e., we have $r = rxr$.*

Of course, any invertible element $r \in U(R)$ is regular, and hence as a special case of the above corollary we deduce that, for any $m \in M$ and $u \in U(R)$, $mu\widehat{\{u^{-1}\}} \subseteq \widehat{\{m\}}$.

We say that an element $m \in M_R$ is an *atom* if the only elements $r \in R$ dividing $m$ are the units elements of $R$. For atoms we have the following result.

**Corollary 2.12.** *If $p \in M$ is an atom, then we have $p(U(R))' = \widehat{\{p\}}$.*

*Proof.* Thanks to Example 2.2(4), we know that $p(U(R))' \subseteq \widehat{\{p\}}$. On the other hand, if $y \overset{1}{\sim} p$, then we derive that there exist $s, t, p' \in R$ such that $y = p'st$ and $p = p'ts$. Since the only divisors of $p$ are invertible elements, we get $y = p'st = p[s^{-1}, t^{-1}]$. This implies that $\widehat{\{p\}} \subseteq p(U(R))'$, and the proof is over. □

Of course, if the base ring $R$ of the module $M_R$ is a commutative ring $R$, every element of $M$ is commutatively closed, and hence any element of $m \in M$ satisfies $m(U(R))' = \widehat{\{m\}}$. In particular, if $M_R = R_R$, the statement of the corollary is not a characterization of atoms.

## 3. Symmetric groups and factorizations

Let us start this section with a list of definitions and notations that will be useful.

**Definitions 3.1.** *Let $R$ be a ring with a unity $1 = 1_R$ and $a, b \in R$. We write:*

*(i)* (a) $a \overset{c}{\sim}_1 b$ *if there exist $x, y \in R$ such that $a = xy$ and $b = yx$.*
(b) $a \overset{*}{\sim}_1 b$ *if there exist $x, y, z \in R$ such that $a = xyz$ and $b = xzy$.*
(c) $a \widehat{\sim}_1 b$ *if there exist $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in R$ and $\pi \in S_n$ such that $a = x_1 x_2 \cdots x_n$ and $b = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$.*

*(ii) For $s \in \{c, *, \wedge\}$ we define:*
(a) $a \overset{s}{\sim}_l b$ *if there exist $x_1, x_2, \ldots, x_l \in R$ such that*

$$a \overset{s}{\sim}_1 x_1 \overset{s}{\sim}_1 x_2 \overset{s}{\sim}_1 \cdots \overset{s}{\sim}_1 x_l = b.$$

(b) $a \overset{s}{\sim} b$ *if there exists $l \in \mathbb{N}$ such that $a \overset{s}{\sim}_l b$.*
(c) $\{a\}^s = \{b \in R \mid a \overset{s}{\sim} b\}$*. We also write $\overline{\{a\}}$ for $\{a\}^c$ and $\widehat{\{a\}}$ for $\{a\}^\wedge$.*

The relation $\overset{c}{\sim}$ was studied in [1] and [2].

**Lemma 3.2.** *Let $a, b$ be elements in a unital ring $R$. Then*

(i) *$a \widehat{\sim} b$ if and only if $a \overset{*}{\sim} b$. Moreover, $\widehat{\sim}$ is an equivalence relation on $R$.*
(ii) *For any $a \in R$, we have $\bigcup_{l \geq 1} \{b \mid a \widehat{\sim}_l b\} = \widehat{\{a\}}$.*
(iii) *If $a \widehat{\sim}_1 a'$ and $b \widehat{\sim}_1 b'$, then $aa' \widehat{\sim}_1 bb'$.*

*Proof.* (i) Suppose first that $a \overset{*}{\sim} b$. Then $b$ is symmetrically related to $a$ and there exists a sequence of elements $a = a_0, \ldots, a_l = b$ in $R$ such that $a = a_0 \overset{*}{\sim}_1 a_1 \overset{*}{\sim}_1 a_2 \overset{*}{\sim}_1 \ldots \overset{*}{\sim}_1 a_l = b$. It is clear that, for $x, y \in R$, $x \overset{*}{\sim}_1 y$ implies that $x \widehat{\sim}_1 y$. This gives that $a \widehat{\sim} b$, as required.

For the converse implication, it is sufficient to prove that if $a \widehat{\sim}_1 b$, then $b$ is symmetrically related to $a$. To do this, assume we have $a = a_1 a_2 \cdots a_n$, $\pi \in S_n$, and $b = a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(n)}$. Since $\pi$ is a product of transpositions, we see that it is enough to show for an arbitrary transposition $\tau$ we have $a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(n)}$ is symmetrically related to $a$. Write $\tau = (ij)$ for $1 \leq i < j \leq n$. We thus get successively:

$$\begin{aligned} a = a_1 \cdots a_i \cdots a_j \cdots a_n &\overset{*}{\sim}_1 a_1 \cdots a_{i-1} a_j a_{j+1} \cdots a_n a_i \cdots a_{j-1} \\ &\overset{*}{\sim}_1 a_1 \cdots a_{i-1} a_j a_i \cdots a_{j-1} a_{j+1} \cdots a_n \\ &\overset{*}{\sim}_1 a_1 \cdots a_{i-1} a_j a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_n a_i \\ &\overset{*}{\sim}_1 a_1 \cdots a_{i-1} a_j a_{i+1} \cdots a_{j-1} a_i a_{j+1} \cdots a_n \\ &= a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(n)}. \end{aligned}$$

This yields the conclusion.
The fact that $\widehat{\sim}$ is an equivalence is a direct consequence of the fact that $\sim$ is an equivalence relation (cf. Proposition 2.4).

(ii) The inclusion $\bigcup_{l\geq 1}\{b \mid a\widehat{\sim}_l b\} \subseteq \widehat{\{a\}}$ is a direct consequence of the statement (i) above. The reverse inclusion is routine since $a \overset{*}{\sim}_1 b$ implies that $a\widehat{\sim}_1 b$.

(iii) Assume that $a\widehat{\sim}_1 a'$ and $b\widehat{\sim}_1 b'$. This implies that there exist $n, m \in \mathbb{N}$, $\pi \in S_n$, $\sigma \in S_m$, $a_1, \ldots, a_n \in R$, and $b_1, \ldots, b_m \in R$ such that $a = a_1 \cdots a_n$, $a' = a_{\pi(1)} \cdots a_{\pi(n)}$, $b = b_1 \cdots b_m$, and $b' = b_{\sigma(1)} \cdots b_{\sigma(m)}$. Define $\delta \in S_{n+m}$ is given by $\delta(i) := \pi(i)$ for all $i = 1, \ldots, n$, and $\delta(i) := \sigma(i-n)$ for all $i = n+1, \ldots, n+m$. We thus obtain $ab = a_1 \cdots a_n b_1 \cdots b_m \widehat{\sim}_1 a_{\delta(1)} \cdots a_{\delta(n)} b_{\delta(n+1)} \cdots b_{\delta(n+m)} = a'b'$, and the proof is done. □

**Definition 3.3.** *Let $S$ be a subset of a ring $R$. We define the symmetric closure of $S$ as $\widehat{S} = \bigcup_{s\in S} \widehat{\{s\}}$.*

It is worth noticing that for an element $x \in R$, the symmetric closure $\widehat{\{x\}}$ is the equivalence class of $x$ corresponding to the equivalence relation $\widehat{\sim}$ defined on $R$.

**Definition 3.4.** *Let $T$ be a subset of a ring $R$. We define recursively a collection of subsets $T_i \subseteq R$, $i \geq 0$, containing $T$ as follows: $T_0 = T$ and for any $i \geq 0$,*

$$\begin{aligned} T_{i+1} = \{x \in R \mid \exists \ell\ \exists a_1, \ldots, a_\ell \in R,\ \exists \pi \in S_\ell \text{ with} \\ x = a_1 \cdots a_\ell \text{ and } a_{\pi(1)} \cdots a_{\pi(\ell)} \in T_i\}. \end{aligned}$$

*It is routine to see that $\widehat{T} = \bigcup_{i\geq 0} T_i$.*

**Lemma 3.5.** *Assume that $T$ is a non-empty subset of a ring $R$. Then the following statements hold:*

- (i) *The chain $T_i$, where $i \geq 0$, is ascending.*
- (ii) *For all $n, m \in \mathbb{N}$, we have $(T_n)_m = T_{n+m}$. In addition, if $T_n = T_{n+1}$, then $T_n = T_{n+k}$ for any $k \geq 0$, and also $\widehat{T} = T_n$.*
- (iii) *$\widehat{T}$ is symmetrically closed, that is, $\widehat{\widehat{T}} = \widehat{T}$.*

*Proof.* (i) Fix $i \geq 0$, and pick an arbitrary element $x$ in $T_i$. Let $x = a_1 \cdots a_\ell$, where $a_1, \ldots, a_\ell \in R$. Then by considering $\pi = id$, where $id$ denotes the identity permutation in $S_\ell$, we obtain $x \in T_{i+1}$. Hence, $T_i \subseteq T_{i+1}$ for all $i \geq 0$.

(ii) We prove this claim by using induction on $m$. Let $m = 1$. It follows readily from the definition that

$$\begin{aligned} (T_n)_1 = &\{x \in R \mid \exists \ell\ \exists a_1, \ldots, a_\ell \in R,\ \exists \pi \in S_\ell \text{ with} \\ &x = a_1 \cdots a_\ell \text{ and } a_{\pi(1)} \cdots a_{\pi(\ell)} \in T_n\} \\ = &T_{n+1}. \end{aligned}$$

Hence, the claim holds for $m = 1$. Now, suppose, inductively, that $m > 0$ and that the result has been shown for $m$, that is to say, $(T_n)_m = T_{n+m}$. Once again, one can conclude rapidly from the definition that

$$\begin{aligned} (T_n)_{m+1} = &\{x \in R \mid \exists \ell\ \exists a_1, \ldots, a_\ell \in R,\ \exists \pi \in S_\ell \text{ with} \\ &x = a_1 \cdots a_\ell \text{ and } a_{\pi(1)} \cdots a_{\pi(\ell)} \in (T_n)_m\} \\ = &\{x \in R \mid \exists \ell\ \exists a_1, \ldots, a_\ell \in R,\ \exists \pi \in S_\ell \text{ with} \\ &x = a_1 \cdots a_\ell \text{ and } a_{\pi(1)} \cdots a_{\pi(\ell)} \in T_{n+m}\} \\ = &T_{n+m+1}. \end{aligned}$$

This completes the inductive step, and hence the claim has been proven by induction. The last assertion is an immediate consequence of the first assertion and part (i).

(iii) Since $\widehat{T} \subseteq \widehat{\widehat{T}}$, it is enough to show the reverse inclusion. To do this, take an arbitrary element $x$ in $\widehat{\widehat{T}}$. This implies that $x \in \widehat{\{t\}}$ for some $t \in \widehat{T}$, and so $t \in \widehat{\{s\}}$ for some $s \in T$. We thus get $x \widehat{\sim} t \widehat{\sim} s$, and hence $x \in \widehat{\{s\}} \subseteq \widehat{T}$. This finishes the proof. □

Recall that for a subset $T$ of a ring $R$, we define $r(T) = \{x \in R \mid Tx = 0\}$, $l(T) = \{x \in R \mid xT = 0\}$, $r(a) = \{x \in R \mid ax = 0\}$, and $l(a) = \{x \in R \mid xa = 0\}$.

The next proposition specifies some particular subsets of the closure of $\widehat{T}$.

**Proposition 3.6.** *Assume that $T$ is a subset of a ring $R$. Then the following statements hold:*

(i) $T^U = \{utu^{-1} \mid u \in U(R), t \in T\} \subseteq T_1$.
(ii) *For all $n \geq 1$, we have $(1 + r(T))^n T \cup T(1 + l(T))^n \subseteq T_n$.*

*Proof.* (i) Let $t \in T$ and $u \in U(R)$. Since $t = u^{-1}ut \in T$, this implies that $utu^{-1} \in T_1$. We therefore get $T^U \subseteq T_1$, as claimed.

(ii) We argue by induction on $n$. Let $n = 1$. Since $r(T)$ (respectively, $l(T)$), one can check that $T(1 + r(T)) \subseteq T$ (respectively, $(1 + l(T))T \subseteq T$). This implies that $(1 + r(T))T \subseteq T_1$ (respectively, $T(1 + l(T)) \subseteq T_1$). Hence, we get $(1 + r(T))T \cup T(1 + l(T)) \subseteq T_1$. Now, suppose, inductively, that $n > 0$ and that the result has been shown for $n$, that is to say, $(1 + r(T))^n T \cup T(1 + l(T))^n \subseteq T_n$. This yields that $(1 + r(T))^n T \subseteq T_n$ (respectively, $T(1 + l(T))^n \subseteq T_n$). Accordingly, one has $(1 + r(T))^n T(1 + r(T)) \subseteq (1 + r(T))^n T \subseteq T_n$ (respectively, $(1 + l(T))T(1 + l(T))^n \subseteq T(1 + l(T))^n \subseteq T_n$). We thus obtain $(1 + r(T))^{n+1} T \subseteq T_{n+1}$ (respectively, $T(1 + l(T))^{n+1} \subseteq T_{n+1}$). This completes the inductive step, and so the claim has been proven by induction. □

**Proposition 3.7.** *Let $a$ be a symmetrically closed element in a ring $R$. Then the following statements hold:*

(i) $r(a) = l(a)$.
(ii) *Let $e$ be an idempotent element in $R$ such that $ae = ea = a$. Then $a$ is a symmetrically closed element in $eRe$.*

*Proof.* (i) We first show that $r(a) \subseteq l(a)$. To do this, let $x \in r(a)$. Hence, $a = a(1 + x)$. Since $a$ is symmetrically closed, we get $a = (1 + x)a$, and so $xa = 0$. This implies that $x \in l(a)$. Accordingly, $r(a) \subseteq l(a)$. Conversely, consider an arbitrary element $x \in l(a)$. This gives that $a = (1 + x)a$. Similarly, one has $a = a(1 + x)$, and thus $ax = 0$. Consequently, $x \in r(a)$. This yields that $l(a) \subseteq r(a)$, as required.

(ii) This assertion is a direct consequence of the definitions. □

**Proposition 3.8.** *Assume that $T$ is a subset of a ring $R$. Then the following statements hold:*

(i) *If $T$ is symmetrically closed, then its complement $R \setminus T$ is symmetrically closed.*
(ii) *Any union (respectively, intersection) of symmetrically closed sets is symmetrically closed.*

(iii) *The collection of symmetrically closed subsets defines a topology on the ring $R$. For this topology, the open sets are also closed.*
(iv) *If $A \subseteq B$ are two subsets of $R$, then $\widehat{A} \subseteq \widehat{B}$.*
(v) *If $T_\lambda \subseteq R$, where $\lambda \in \Lambda$, are subsets of a ring $R$, then*
  (a) $\cup_{\lambda\in\Lambda}\widehat{T_\lambda} = \widehat{\cup_{\lambda\in\Lambda}T_\lambda}$.
  (b) $\widehat{\cap_{\lambda\in\Lambda}T_\lambda} \subseteq \cap_{\lambda\in\Lambda}\widehat{T_\lambda}$.

*Proof.* (i) This is an obvious consequence of the fact that $\widehat{\sim}$ is an equivalence relation (cf. Lemma 3.2(i)).

(ii) Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a collection of symmetrically closed sets in $R$. Select an arbitrary element $x$ in $\cup_{\lambda\in\Lambda}T_\lambda$. This implies that $x \in T_\lambda$ for some $\lambda \in \Lambda$. since $T_\lambda$ is symmetrically closed, one has $(T_\lambda)_1 = T_\lambda$. Hence, there exist $a_1, \ldots, a_\ell \in R$ and $\pi \in S_\ell$ with $x = a_1 \cdots a_\ell$ and $a_{\pi(1)} \cdots a_{\pi(\ell)} \in T_\lambda$. We thus have $a_{\pi(1)} \cdots a_{\pi(\ell)} \in \cup_{\lambda\in\Lambda}T_\lambda$, and so $x \in (\cup_{\lambda\in\Lambda}T_\lambda)_1$. This yields that $\cup_{\lambda\in\Lambda}T_\lambda \subseteq (\cup_{\lambda\in\Lambda}T_\lambda)_1$. Based on Lemma 3.5(ii), one can deduce that $\cup_{\lambda\in\Lambda}T_\lambda$ is symmetrically closed. To show the symmetrically closedness of $\cap_{\lambda\in\Lambda}T_\lambda$, one should consider part (i) and the fact that $\cap_{\lambda\in\Lambda}T_\lambda = R \setminus (\cup_{\lambda\in\Lambda}(R \setminus T_\lambda))$.

(iii) Since $R$ itself is symmetrically closed, it follows from part (i) that the empty set is also symmetrically closed. Now, the claim can be concluded from parts (i) and (ii).

(iv) This assertion is an immediate consequence of the facts that $\widehat{A} = \bigcup_{a\in A}\widehat{\{a\}}$ and $\widehat{B} = \bigcup_{b\in B}\widehat{\{b\}}$.

(v) On account of Lemma 3.5(iii), for any $\lambda \in \Lambda$, $\widehat{T_\lambda}$ is symmetrically closed. It follows now from part (ii) that $\cup_{\lambda\in\Lambda}\widehat{T_\lambda}$ is symmetrically closed. Since, for any $\lambda \in \Lambda$, $T_\lambda \subseteq \cup_{\lambda\in\Lambda}\widehat{T_\lambda}$, using part (iv) gives that $\widehat{\cup_{\lambda\in\Lambda}T_\lambda} \subseteq \cup_{\lambda\in\Lambda}\widehat{T_\lambda}$. Conversely, we note that, for any $\lambda \in \Lambda$, parts (ii) and (iv) imply that $\widehat{T_\lambda} \subseteq \widehat{\cup_{\lambda\in\Lambda}T_\lambda}$, and hence $\cup_{\lambda\in\Lambda}\widehat{T_\lambda} \subseteq \widehat{\cup_{\lambda\in\Lambda}T_\lambda}$. This proves (a).

Because, for any $\lambda \in \Lambda$, $\widehat{T_\lambda}$ is symmetrically closed, part (ii) yields that $\cap_{\lambda\in\Lambda}\widehat{T_\lambda}$ is symmetrically closed as well. Due to $\cap_{\lambda\in\Lambda}T_\lambda \subseteq \widehat{T_\lambda}$ for any $\lambda \in \Lambda$, using part (iv) implies that $\widehat{\cap_{\lambda\in\Lambda}T_\lambda} \subseteq \cap_{\lambda\in\Lambda}\widehat{T_\lambda}$. This proves (b). □

**Theorem 3.9.** *Assume that $\varphi : R \longrightarrow W$ is a ring homomorphism. Then the following statements hold:*

(i) *For any $X \subseteq R$, $\varphi(\widehat{X}) \subseteq \widehat{\varphi(X)}$.*
(ii) *If $\varphi$ is a ring isomorphism, then for any $X \subseteq R$, $\varphi(\widehat{X}) = \widehat{\varphi(X)}$.*
(iii) *If $T \subseteq W$ is symmetrically closed in $W$, then $\varphi^{-1}(T)$ is symmetrically closed in $R$.*

*Proof.* (i) Since $\widehat{X} = \bigcup_{i\geq 0} X_i$ and $\widehat{\varphi(X)} = \bigcup_{i\geq 0}\varphi(X)_i$, and in view of Lemma 3.5(i), it is sufficient for us to show that $\varphi(X_1) \subseteq \varphi(X)_1$. To do this, choose an arbitrary element $y \in \varphi(X_1)$. Hence, $y = \varphi(z)$ for some $z \in X_1$. This yields that there exist $a_1, \ldots, a_\ell \in R$ and $\pi \in S_\ell$ with $z = a_1 \cdots a_\ell$ and $a_{\pi(1)} \cdots a_{\pi(\ell)} \in X$. We thus get $y = \varphi(z) = \varphi(a_1) \cdots \varphi(a_\ell)$ and $\varphi(a_{\pi(1)}) \cdots \varphi(a_{\pi(\ell)}) = \varphi(a_{\pi(1)} \cdots a_{\pi(\ell)}) \in \varphi(X)$. This gives rise to $y \in \varphi(X)_1$, as required.

(ii) Due to Lemma 3.5(ii) and part (i), one has to demonstrate that $\varphi(X)_1 \subseteq \varphi(X_1)$. For this purpose, let $y \in \varphi(X)_1$. This implies that there exist $a_1, \ldots, a_\ell \in W$ and $\pi \in S_\ell$ with $y = a_1 \cdots a_\ell$ and $a_{\pi(1)} \cdots a_{\pi(\ell)} \in \varphi(X)$. Hence, $a_{\pi(1)} \cdots a_{\pi(\ell)} =$

$\varphi(z)$ for some $z \in X$. Because $a_{\pi(i)} \in W$, for $i = 1, \ldots, \ell$, and $\varphi$ is surjective, this yields that there exist $r_1, \ldots, r_\ell \in R$ such that $a_{\pi(i)} = \varphi(r_i)$ for $i = 1, \ldots, \ell$. Consequently, we obtain $\varphi(z) = a_{\pi(1)} \cdots a_{\pi(\ell)} = \varphi(r_1 \cdots r_\ell)$. Since $\varphi$ is injective, one can derive that $z = r_1 \cdots r_\ell \in X$. Hence, $r_1 \cdots r_\ell \in X_1$. Finally, we note that $y = \varphi(z) = \varphi(r_1 \cdots r_\ell) \in \varphi(X_1)$. This finishes our argument.

(iii) According to Lemma 3.5(ii), it is enough to verify that $(\varphi^{-1}(T))_1 \subseteq \varphi^{-1}(T)$. Take an arbitrary element $z$ in $(\varphi^{-1}(T))_1$. This gives that there exist $a_1, \ldots, a_\ell \in R$ and $\pi \in S_\ell$ with $z = a_1 \cdots a_\ell$ and $a_{\pi(1)} \cdots a_{\pi(\ell)} \in \varphi^{-1}(T)$. We thus gain $\varphi(a_{\pi(1)}) \cdots \varphi(a_{\pi(\ell)}) = \varphi(a_{\pi(1)} \cdots a_{\pi(\ell)}) \in T$. On account of $T \subseteq W$ is symmetrically closed, this leads to $\varphi(a_1 \cdots a_\ell) = \varphi(a_1) \cdots \varphi(a_\ell) \in T$. Accordingly, one has $z = a_1 \cdots a_\ell \in \varphi^{-1}(T)$, and the proof is complete. □

We now collect a few properties in the following proposition. Recall that a subset $S$ of a ring $R$ is *multiplicatively closed* whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$ too. Following [1], let us recall for $a, b \in R$, we say $d_c(a, b) = n$, also denoted by $a \overset{c}{\sim}_n b$, if there exists $x_1, \ldots, x_n, y_1, \ldots, y_n \in R$ such that

$$a = x_1 y_1, y_1 x_1 = x_2 y_2, y_2 x_2 = x_3 y_3, \ldots, y_{n-1} x_{n-1} = x_n y_n, y_n x_n = b.$$

In addition, the *commutative closure* of $a \in R$ is given by

$$\overline{\{a\}} = \{b \in R \mid \exists n \in \mathbb{N} \cup \{0\} \text{ with } d_c(a, b) = n\}.$$

For $S \subset R$, we define $\overline{S} = \bigcup_{s \in S} \overline{\{s\}}$.

**Proposition 3.10.** *Let $a$ and $b$ be elements in a ring $R$, and $S$ a subset of $R$. Then the following statements hold:*

(i) $\overline{\{a\}} \subseteq \widehat{\{a\}}$.
(ii) *Any symmetrically closed set is commutatively closed.*
(iii) $\widehat{\{ab\}} = \widehat{\{ba\}}$.
(iv) $a\overline{\{b\}} \subseteq a\widehat{\{b\}} \subseteq \overline{\{a\}}\widehat{\{b\}} \subseteq \widehat{\{a\}}\widehat{\{b\}} \subseteq \widehat{\{ab\}}$.
(v) *Let $B$ denote $1 - r(a)$, where $r(a) = \{x \in R \mid ax = 0\}$ stands for the right annihilator of the element $a$. Then $\widehat{\{Ba\}} \subseteq \widehat{\{a\}}$.*
(vi) *If $S$ is a multiplicatively closed set, then $\widehat{S}$ is multiplicatively closed as well.*

*Proof.* (i) The first inclusion is clear since $1 \in R$ and $1.ab \overset{\frown}{\sim}_1 1.ba$.

(ii) This is a direct consequence of part (i) above.

(iii) We have $ab \in \widehat{\{ba\}}$, and hence $\widehat{\{ab\}} \subseteq \widehat{\{ba\}}$. A similar argument gives $\widehat{\{ba\}} \subseteq \widehat{\{ab\}}$.

(iv) The inclusion $\overline{\{b\}} \subseteq \widehat{\{b\}}$ (cf. part (i) above) implies that $a\overline{\{b\}} \subseteq a\widehat{\{b\}}$ and $\overline{\{a\}}\widehat{\{b\}} \subseteq \widehat{\{a\}}\widehat{\{b\}}$. Similarly, the second inclusion follows from the fact that $a \in \overline{\{a\}}$. The last inclusion $\widehat{\{a\}}\widehat{\{b\}} \subseteq \widehat{\{ab\}}$ is due to the fact that permuting the factors of the first element, say $a$, of a product $ab$ is also permuting a factorization of $ab$.

(v) This is due to the fact that $\{a\} = aB$, and hence $Ba \subseteq \overline{\{a\}} \subseteq \widehat{\{a\}}$. We therefore have $\widehat{\{Ba\}} \subseteq \widehat{\{a\}}$.

(vi) Let $x, y \in \widehat{S}$. Thus, we have elements $s, t \in S$ such that $x \in \widehat{\{s\}}$ and $y \in \widehat{\{t\}}$. The fourth inclusion in part (iv) above then gives $xy \in \widehat{\{s\}}\widehat{\{t\}} \subseteq \widehat{\{st\}}$. Since $S$ is multiplicatively closed, this implies that $xy \in \widehat{S}$ . □

**Definition 3.11.** [2, Definition 2.1] *An element $a \in R$ is said to be commutatively closed when $\overline{\{a\}} = \{a\}$. If $a, b \in R$ we say that $b$ is a factor of $a$ if there exist elements $c, d \in R$ such that $a = cbd$.*

**Proposition 3.12.** *Let $R$ be a unital ring, and $z \in R$ be a symmetrically closed element. Then the following statements hold:*

(i) *The element $z$ commutes with units.*
(ii) *If 2 is not a zerodivisor in $R$, then $z$ commutes with idempotent elements.*
(iii) *The element $z$ commutes with its factors.*

*Proof.* The claim can be deduced from [2, Proposition 2.2] and Proposition 3.10(i). □

**Corollary 3.13.** *Let $z \in R$ be a symmetrically closed element. Then the following statements hold:*

(i) *If $z$ is nilpotent, then $RzR$ is a nilpotent ideal.*
(ii) *If $z$ is not a right (or left) zerodivisor, then $R$ is Dedekind-finite.*

*Proof.* The assertion is an immediate consequence of [2, Corollary 2.3] and Proposition 3.10(i). □

The definition below is sometimes useful especially while making explicit computations. It is similar to the one given in Definition 3.4.

**Definitions 3.14.** *Let $S \subset R$ be a nonempty subset of $R$. We define, for $s \in \{c, *, \wedge\}$,*

$$S_n^s = \{x \in R \mid \exists\, x_0 \in S \text{ such that } x \overset{s}{\sim}_n x_0 \}.$$

*In particular, for any $s \in S$, we have $\{s\}^* = \bigcup_{n \geq 0} \{s\}_n^*$.*

Recall that a ring $R$ is *Dedekind-finite* if, for any $a, b \in R$, we have $ab = 1$ implies $ba = 1$. When the ring $R$ is Dedekind-finite, we can describe the symmetric closure of any subset $S$ contained in $R$.

**Proposition 3.15.** *Let $S \subseteq R$ be a subset of a Dedekind-finite ring $R$. Then the following statements hold:*

(i) *If $S$ is a group, then $\widehat{S}$ is a group as well.*
(ii) *$\{1\}_n^*$ is the set of products of at most $n$ commutators.*
(iii) *The closed set $\widehat{\{1\}}$ is the derived group $U(R)'$ of the group of units of $R$.*
(iv) *If $S \subseteq U(R)$, then $\widehat{S} = S(U(R))'$.*

*Proof.* (i) Since $R$ is Dedekind-finite, any factor of an invertible element is also invertible. The fact that $S$ is a group then implies that the elements of $S_1^*$ are invertible. Moreover, if $x \in S_1^*$, there exists $s \in S$ such that $x \widehat{\sim}_1 s$, and hence $x^{-1} \widehat{\sim}_1 s^{-1}$. Since $S$ is a group, we thus get $x^{-1} \in S_1^*$. Similarly, for any $l \in \mathbb{N}$, the elements of $S_l^*$ are invertible with inverses in $S_l^*$. Let $a, b \in \widehat{S}$. So, there exists $l \in \mathbb{N}$ such that $a, b \in S_l^*$, and since $b^{-1} \in S_l^* \subseteq \widehat{S}$, we get $ab^{-1} \in \widehat{S}$. This shows that $\widehat{S}$ is a group.

(ii) First remark that $x$ and all the elements that appear in a path linking 1 and $x$ are invertible (in fact, $\widehat{\{1\}}$ is a group). The result is thus clear since for invertible elements $a, b, c \in U(R)$, we have $abc = acb[b^{-1}, c^{-1}]$.

(iii) Clearly, $\widehat{\{1\}}$ is a subgroup of $U(R)$, and by part (ii) above, we get that in fact $\widehat{\{1\}} \subseteq U(R)'$, the derived subgroup of $U(R)$. On the other hand, any commutator is an element of $\{1\}_1^*$, and this yields the assertion.

(iv) In the light of Proposition 3.10(iv), we have $S \subseteq S\widehat{\{1\}}$. Because the elements of $S$ are invertible, we also get the reverse inclusion. $\square$

**Corollary 3.16.** *Let $R$ be a ring. Then the following statements are equivalent:*

(i) *$R$ is Dedekind-finite.*
(ii) $\overline{\{1\}} = \{1\}$.
(iii) $\widehat{\{1\}} = U(R)'$.

*Moreover, when $R$ is Dedekind-finite, we have for any $a \in U(R)$, $\widehat{\{a\}} = a\widehat{\{1\}}$.*

*Proof.* We know that the set $\{1\}$ is commutatively closed (i.e., $\overline{\{1\}} = \{1\}$) if and only if $R$ is Dedekind-finite. Thus, (i) and (ii) are equivalent. In addition, Proposition 3.15(iii) gives the implication (i) $\Rightarrow$ (iii). To finish the argument, one has to show the implication (iii) $\Rightarrow$ (i). To see this, let $ab = 1$, where $a, b \in R$. This yields that $ba \in \overline{\{1\}}$. Since $\overline{\{1\}} \subseteq \widehat{\{1\}} = U(R)' \subseteq U(R)$, we obtain that $ba \in U(R)$. Now, by considering the fact that $(ba)^2 = ba$, we get $ba = 1$. This means that $R$ is Dedekind-finite. $\square$

**Example 3.17.** *Let $\mathbb{H}$ denote the division ring of real quaternions. For $x = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ we define $N(x) = a_0^2 + a_1^2 + a_3^2 + a_3^2$. Moreover, let $\Gamma := \{x \in \mathbb{H} : N(x) = 1\}$. Then $\widehat{\{1\}} = \Gamma$.*

*Proof.* In the light of [12, Proposition 1.3.5] and Corollary 3.16(iii), one can rapidly conclude that $\widehat{\{1\}} = U(\mathbb{H})'$, and hence $\widehat{\{1\}} = \Gamma$. $\square$

In a similar way, we can now look at the symmetric closure of $\{0\}$ when $R$ is *reversible*, i.e., when $\overline{\{0\}} = \{0\}$. Recall first that in any ring $R$, the set $N(R)$ of nilpotent elements is always commutatively closed. Of course, $N(R)$ is not symmetrically closed in general as we can see by considering the case of a matrix ring $R = M_n(k)$, $n \geq 2$ over a field $k$. In fact, in this case, we have $\widehat{N(R)} = \widehat{\{0\}}$ is the set of all singular matrices (cf. Corollary 3.21).

We recall that a ring $R$ is *semi-commutative* if for any $a, b \in R$ we have that $ab = 0$ implies $aRb = 0$. Furthermore, any reversible ring is semi-commutative.

**Proposition 3.18.** *If $R$ is semi-commutative, then $N(R)$ is symmetrically closed. In particular, this holds if $R$ is reversible.*

*Proof.* It is enough to show that, if $a \in N(R)$ and $b \overset{*}{\sim}_1 a$, then $b \in N(R)$. To do this, assume that $a = xyz$ and $b = xzy$. Let also $n \in \mathbb{N}$ be such that $a^n = 0$. Since $R$ is semi-commutative, for any $r_1, r_2, \ldots, r_{3n} \in R$, we have $xr_1yr_2zr_3xr_4 \cdots r_{3n-1}zr_{3n} = 0$. Replacing $r_{1+3l}, r_{2+3l}, r_{3+3l}$ $(l \geq 0)$ by $z, x, y$ respectively, we conclude that $b^{2n} = 0$. Accordingly, one has $b \in N(R)$, as required. $\square$

Recall that a ring $R$ is said to be *symmetric* if and only if for any $a, b, c \in R$, $abc = 0$ implies $acb = 0$.

Looking at the symmetric closure of $\{0\}$ we easily get the following proposition:

**Proposition 3.19.** $\widehat{\{0\}} = \{0\}$ *if and only if $R$ is symmetric.*

*Proof.* (⇒) Let $abc = 0$, where $a, b, c \in R$. Due to $abc \overset{*}{\sim}_1 acb$, this leads to $acb \in \widehat{\{0\}}$, and so $acb \in \{0\}$. Hence, $acb = 0$, that is, $R$ is symmetric.

(⇐) We must show that $\{0\}_1 \subseteq \{0\}$ by virtue of Lemma 3.5(ii). To see this, pick an arbitrary element $z$ in $\{0\}_1$. This says that there exist $a_1, \ldots, a_\ell \in R$ and $\pi \in S_\ell$ with $z = a_1 \cdots a_\ell$ and $a_{\pi(1)} \cdots a_{\pi(\ell)} \in \{0\}$. Hence, we get $a_{\pi(1)} \cdots a_{\pi(\ell)} = 0$. Since $R$ is symmetric, and considering the fact that $\pi$ is a product of transpositions, we can mimic the discussion which has been stated in Lemma 3.2(i). This gives rise to $z = a_1 \cdots a_\ell = 0$, and so $z \in \{0\}$, as required. □

The next theorem determines the symmetric classes in case of a matrix ring over a division ring. Note that $GL_n(R)$ denotes the general linear group of non-singular $n \times n$ matrices with entries in $R$.

**Theorem 3.20.** *Let $D$ be a division ring, $n \in \mathbb{N}$, and $A \in M_n(D)$. Also, let 1 denote the identity matrix. Then the following statements hold:*

(i) $\widehat{\{1\}} = GL_n(D)'$.
(ii) *If $A \in GL_n(D)$, then $\widehat{A} = A\widehat{\{1\}}$.*
(iii) *If $A$ is singular, then $\widehat{A} = \widehat{\{0\}}$.*

*Proof.* (i) First note that the ring $R = M_n(D)$ is Dedekind-finite. It follows now from Corollary 3.16(iii) that $\widehat{\{1\}} = GL_n(D)'$.

(ii) This assertion can be deduced from Proposition 3.15(iv).

(iii) We first show that $\widehat{E} = \widehat{\{0\}}$ for any idempotent matrix $E = E^2$ such that $\operatorname{rank}(E) = r < n$. The proof proceeds by induction on $r$. If $r = 0$, we have $E = 0$. Suppose that the property is proven for idempotent matrices of rank $0 \le s < r$ and consider an idempotent matrix $E$ of rank $r$. Let us denote $F_r$ the idempotent matrix $F_r = \operatorname{diag}(I_r, 0_{n-r})$. There exists an invertible matrix $P \in GL_n(D)$ such that $PEP^{-1} = F_r$. In particular, we have $\widehat{E} = \widehat{F_r}$. If $Q \in GL_n(D)$ is the permutation matrix associated to the transposition $(r, r+1)$, we get $QF_rQ^{-1}$ is the matrix $\operatorname{diag}(I_{r-1}, 0, 1, 0, \ldots, 0)$, and hence $F_r = F_r^2 \overset{*}{\sim}_1 F_rQF_rQ^{-1} = F_{r-1}$. We thus gain $\widehat{E} = \widehat{F_r} = \widehat{F_{r-1}}$. The induction hypothesis then gives $\widehat{E} = \widehat{F_r} = \widehat{F_{r-1}} = \widehat{\{0\}}$, as claimed. Now, if $A \in M_n(D)$ is a singular matrix, then we can write $A = E_1E_2 \cdots E_r$, where $E_i = E_i^2$ for $i = 1, \ldots, r$ (cf. [9]). Based on Proposition 3.10, we have $0 \in \widehat{\{0\}} = \widehat{E_1} \cdots \widehat{E_r} \subseteq \widehat{A}$. Accordingly, one concludes that $0 \in \widehat{A}$, and hence $\widehat{\{0\}} = \widehat{A}$. This yields the desired result. □

In the next theorem, we will use the Dieudonné determinant. Recall that if $D$ is a division ring and $n \in \mathbb{N}$, the Dieudonné determinant is a map $GL_n(D) \xrightarrow{\text{Det}} \frac{D^*}{U(D)'}$ such that $\operatorname{Det}(AB) = \operatorname{Det}(A)\operatorname{Det}(B)$ and $\operatorname{Det}(I_n) = 1$. The kernel of the map Det is denoted as $SL_n(D)$, and is the subgroup of $GL_n(D)$ generated by the elementary matrices. If $A \in M_n(D)$ is singular, we put $\operatorname{Det}(A) := 0$. If $D$ is commutative, then Det is the usual determinant for matrices with coefficients in a field. The interested reader might consult [6] for more information.

**Corollary 3.21.** *Let $R = M_n(D)$ be a matrix ring over a division ring $D$. Then for any $A \in R$, we have $\widehat{\{A\}} = \{B \in R \mid \operatorname{Det}(B) = \operatorname{Det}(A)\}$.*

*Proof.* If $B \in \widehat{\{A\}}$, then there exist matrices $A_1, \ldots, A_l \in M_n(D)$ such that

$$A \widehat{\sim}_1 A_1 \widehat{\sim}_1 A_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 A_l = B.$$

Hence, to show that $\mathrm{Det}(A) = \mathrm{Det}(B)$, it is enough to show that if $X, Y \in M_n(D)$ are such that $X \widehat{\sim}_1 Y$, then $\mathrm{Det}(X) = \mathrm{Det}(Y)$. Since the determinant is a multiplicative map, this is clear.

Conversely, suppose that $A, B \in M_n(D)$ are such that $\mathrm{Det}(A) = \mathrm{Det}(B)$. If $\mathrm{Det}(A) = \mathrm{Det}(B) \neq 0$, then both $A$ and $B$ are invertible and we have $\mathrm{Det}(AB^{-1}) = 1$. This means that $AB^{-1} \in SL_n(D)$, and hence we obtain $AB^{-1}$ is a product of multiplicative commutators. In particular, $AB^{-1} \in \widehat{\{1\}}$. Now, if $\mathrm{Det}(A) = \mathrm{Det}(B) = 0$, then by Theorem 3.20(iii), we get $\widehat{\{A\}} = \widehat{\{0\}} = \widehat{\{B\}}$, as claimed. □

**Lemma 3.22.** *Assume that $R$ and $S$ are two rings, and $(r, s) \in R \times S$. Then $\widehat{\{(r, s)\}} = \widehat{\{r\}} \times \widehat{\{s\}}$.*

*Proof.* Take an arbitrary element $(x, y)$ in $\widehat{\{(r, s)\}}$. This yields that $(x, y) \in \widehat{\{(r, s)\}}_n$ for some $n \geq 1$. We thus have there exist elements

$$(r, s) = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (x, y) \in R \times S,$$

such that

$$(r, s) = (x_1, y_1) \widehat{\sim}_1 (x_2, y_2) \widehat{\sim}_1 \cdots \widehat{\sim}_1 (x_n, y_n) = (x, y).$$

This gives rise to $r = x_1 \widehat{\sim}_1 x_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 x_n = x$ and $s = y_1 \widehat{\sim}_1 y_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 y_n = y$. We therefore get $(x, y) \in \widehat{\{r\}} \times \widehat{\{s\}}$. To establish the reverse inclusion, assume that $(x, y) \in \widehat{\{r\}} \times \widehat{\{s\}}$. Hence, one has $x \in \widehat{\{r\}}$ and $y \in \widehat{\{s\}}$. We can conclude that $x \in \widehat{\{r\}}_n$ and $y \in \widehat{\{s\}}_m$ for some $n, m \in \mathbb{N}$. This implies that there exist elements $r = x_1, x_2, \ldots, x_n = x \in R$ and $s = y_1, y_2, \ldots, y_m = y \in S$ such that $x_1 \widehat{\sim}_1 x_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 x_n$ and $y_1 \widehat{\sim}_1 y_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 y_m$. Without loss of generality, one may assume that $m \geq n$. We thus have the following

$$\begin{aligned}(r, s) =& (x_1, y_1) \widehat{\sim}_1 (x_2, y_2) \widehat{\sim}_1 \cdots \widehat{\sim}_1 (x_n, y_n) = (x, y_n) \\ & \widehat{\sim}_1 (x, y_{n+1}) \widehat{\sim}_1 \cdots \widehat{\sim}_1 (x_n, y_m) = (x, y).\end{aligned}$$

Accordingly, we gain $(x, y) \in \widehat{\{(r, s)\}}_m$, and hence $(x, y) \in \widehat{\{(r, s)\}}$, as claimed. □

We are now able to determine the symmetric closure of any elements of an Artinian semisimple ring.

**Theorem 3.23.** *Let $R = \prod_{i=1}^{l} M_{n_i}(D_i)$ be an Artinian semisimple ring, where $D_i$'s are division rings. If $A = (A_1, A_2, \ldots, A_l) \in R$, then*

$$\widehat{A} = \{B = (B_1, B_2, \ldots, B_l) \in R \mid \mathrm{Det}(A_i) = \mathrm{Det}(B_i), i = 1, \ldots, l\}.$$

*Proof.* The proof is direct consequence of Lemma 3.22 and Corollary 3.21. □

## 4. Symmetrically closed graphs and their diameters

In this section, we define a graph structure on each symmetry class. Two notions of distances are defined in these graphs. These distances and the diameters of the graphs are compared. To accomplish this, we start with the following definitions. In addition, we refer the reader to [4] for any definition and terminology concerning graph theory.

**Definitions 4.1.** *Let $R$ be a unital ring $R$ and $s \in \{c, *, \wedge\}$.*

*(1) The elements of a class determined by $\overset{s}{\sim}$ can be seen as the set of vertices of a graph. Two elements $x, y$ in the same class are said to be adjacent if $x \overset{s}{\sim}_1 y$.*

*(2) Let $x, y \in R$ be such that $x \overset{s}{\sim} y$, we define $d_s(x, y) = \min\{n \in \mathbb{N} \mid x \overset{s}{\sim}_n y\}$. We adopt the convention that $d_s(x, x) = 0$. It is not hard to check that $d_s$ is a distance. This distance corresponds to the minimal length of the paths between two elements (vertices) in a class determined by $\overset{s}{\sim}$.*

*(3) For a subset $S$ of $R$, we define*

$$\operatorname{diam}_s(S) = \sup\{d_s(x, y) \mid x, y \in S \text{ and } x \overset{s}{\sim} y\}.$$

**Remarks 4.2.**
1. As was proved in Lemma 3.2 the classes determined by $\overset{*}{\sim}$ and $\overset{\wedge}{\sim}$ are the same. So the vertices of the graphs determined by these relations are the same, but, of course, the paths are different.
2. Lemma 3.2 also shows that if $x, y \in \widehat{\{a\}}$, then $d_\wedge(x, y) \leq d_*(x, y)$.
3. A class $\widehat{\{a\}}$ is a connected graph for both $\overset{*}{\sim}$ and $\overset{\wedge}{\sim}$.
4. Since $R$ is the disjoint union of the classes with respect to $\overset{\wedge}{\sim}$ (or $\overset{*}{\sim}$), the ring $R$ itself can be considered as a graph. With this point of view we have, for $s \in \{\wedge, *\}$, $\operatorname{diam}_s(R) = \sup\{\operatorname{diam}_s(\widehat{\{z\}}) \mid z \in R\}$.

In what follows, we collect some results which are related to above definitions.

**Theorem 4.3.** *Let $R$ be a unital ring. Then the following statements hold:*

(i) *If $t \in \widehat{\{z\}}$, then for any $m \in \mathbb{N}$, $t^m \in \widehat{\{z^m\}}$.*

(ii) *A subset $S$ of $R$ is symmetrically closed and connected if and only if $S = \widehat{\{z\}}$ for some $z \in R$.*

(iii) *For any subset $S$ of $R$, $\operatorname{diam}_\wedge(S) \leq \operatorname{diam}_\wedge(\widehat{S})$ (respectively, $\operatorname{diam}_*(S) \leq \operatorname{diam}_*(\widehat{S})$).*

*Proof.* (i) Let $t \in \widehat{\{z\}}$, and fix $m \in \mathbb{N}$. This implies that $t \in \widehat{\{z\}}_n$ for some $n \geq 0$. If $n = 0$, then $t = z$ and the result is clear. So suppose that $n > 1$ and there exist elements $z = x_1, x_2, \ldots, x_n = t$ such that $x_1 \widehat{\sim}_1 x_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 x_n$. In view of Lemma 3.2(iii), one can conclude that $z^m = x_1^m \widehat{\sim}_1 x_2^m \widehat{\sim}_1 \cdots \widehat{\sim}_1 x_n^m = t^m$. We therefore have $t^m \in \widehat{\{z^m\}}$.

(ii) Since $S$ is closed, it is a union of classes corresponding to the equivalence relation $\widehat{\sim}$. Due to the fact that $S$ is connected, $S$ is in fact equal to a single class, and for any $z \in S$, we have $S = \widehat{\{z\}}$.

(iii) This is clear since $S \subseteq \widehat{S}$ □

**Proposition 4.4.** *Let $S$ be a subset of a ring $R$. Then the following statements hold:*

(i) *$\operatorname{diam}_*(S) \leq \operatorname{diam}_c(S)$. In particular, if $\operatorname{diam}_c(S)$ (respectively, $\operatorname{diam}_*(S)$) is finite (respectively, infinite), then $\operatorname{diam}_*(S)$ (respectively, $\operatorname{diam}_c(S)$) is finite (respectively, infinite).*

(ii) *If $R$ is a non-commutative Dedekind-finite, then $\operatorname{diam}_*(U(R)) = 1$. In particular, if $D$ is a division ring, then $\operatorname{diam}_*(D) = 1$.*

*Proof.* (i) It follows from Proposition 3.10(i) that $d_*(x, y) \leq d_c(x, y)$ for any $x, y \in S$. This leads immediately to $\operatorname{diam}_*(S) \leq \operatorname{diam}_c(S)$.

(ii) By virtue of [1, Lemma 1.10], one has $\operatorname{diam}_c(U(R)) = 1$. Now, part (i) implies that $\operatorname{diam}_*(U(R)) = 1$. To prove the last claim, one can combine part (i) and [1, Proposition 1.11]. □

**Proposition 4.5.** *Assume that $z$ is an element in a ring $R$. If $n \in \mathbb{N}$ is the minimal number such that $\widehat{\{z\}} = \widehat{\{z\}}_n$, then $n \leq \operatorname{diam}_\wedge(\widehat{\{z\}}) \leq 2n$.*

*Proof.* Let $x, y \in \widehat{\{z\}}$. Hence, one obtains $x, y \in \widehat{\{z\}}_n$. This gives that there exist elements $z = x_1, x_2, \ldots, x_n = x \in R$ and $z = y_1, y_2, \ldots, y_n = y \in R$ such that $x_1 \widehat{\sim}_1 x_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 x_n$ and $y_1 \widehat{\sim}_1 y_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 y_n$. This means that $d_\wedge(x, z) \leq n$ and $d_\wedge(y, z) \leq n$. It follows now from the triangle inequality that

$$d_\wedge(x, y) \leq d_\wedge(x, z) + d_\wedge(z, y) \leq 2n,$$

and so $\operatorname{diam}_\wedge(\widehat{\{z\}}) \leq 2n$. Since $n$ is minimal, this implies that $n \leq \operatorname{diam}_\wedge(\widehat{\{z\}})$. This completes the proof. □

**Proposition 4.6.** *Let $R$ and $S$ be two rings. Also, let $\operatorname{diam}_\wedge(R) = n$ and $\operatorname{diam}_\wedge(S) = m$. Then $\operatorname{diam}_\wedge(R \times S) = \max\{n, m\}$.*

*In addition, a similar result holds replacing $\operatorname{diam}_\wedge$ by $\operatorname{diam}_*$.*

*Proof.* It follows from the definition that

$$\operatorname{diam}_\wedge(R \times S) = \sup\{\operatorname{diam}_\wedge(\widehat{(r, s)}) \ : \ (r, s) \in R \times S\},$$

where $\operatorname{diam}_\wedge(\widehat{(r, s)}) := \sup\{d_\wedge((x_1, y_1), (x_2, y_2)) \ : \ (x_1, y_1), (x_2, y_2) \in \widehat{\{(r, s)\}}\}$. Let $\operatorname{diam}_\wedge(R \times S) = \operatorname{diam}_\wedge(G_{(r,s)})$, where $(r, s) \in R \times S$. Hence, there exist elements $(x_1, y_1), (x_2, y_2) \in \widehat{\{(r, s)\}}$ such that $\operatorname{diam}_\wedge(\widehat{(r, s)}) := d_\wedge((x_1, y_1), (x_2, y_2))$. On account of Lemma 3.22, one can derive that $x_1, x_2 \in \widehat{\{r\}}$ and $y_1, y_2 \in \widehat{\{s\}}$. One can deduce from the assumptions $\operatorname{diam}_\wedge(R) = n$ and $\operatorname{diam}_\wedge(S) = m$ that $d_\wedge(x_1, x_2) := t \leq n$ and $d_\wedge(y_1, y_2) := k \leq m$. Accordingly, there exist elements $x_1 = u_1, u_2, \ldots, u_t = x_2 \in R$ and $y_1 = v_1, v_2, \ldots, v_k = y_2 \in S$ such that $u_1 \widehat{\sim}_1 u_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 u_t$ and $v_1 \widehat{\sim}_1 v_2 \widehat{\sim}_1 \cdots \widehat{\sim}_1 v_k$. Without loss of generality, one may assume that $k \geq t$. This leads to the following

$$\begin{aligned}(x_1, y_1) =& (u_1, v_1) \widehat{\sim}_1 (u_2, v_2) \widehat{\sim}_1 \cdots \widehat{\sim}_1 (u_t, v_t) = (x_2, v_t) \\ & \widehat{\sim}_1 (x_2, v_{t+1}) \widehat{\sim}_1 \cdots \widehat{\sim}_1 (x_2, v_k) = (x_2, y_2).\end{aligned}$$

Therefore, one derives that $d_\wedge((x_1, y_1), (x_2, y_2)) \leq \max\{n, m\}$. This implies that $\operatorname{diam}_\wedge(R \times S) = \max\{n, m\}$, as desired. □

**Theorem 4.7.** *Let $D$ be a division ring and $n \in \mathbb{N}$. Then $\operatorname{diam}_\wedge(M_n(D)) \leq 2$.*

*Proof.* Let $A, B \in M_n(D)$ be matrices such that $B \in \widehat{A}$. According to Corollary 3.21, we derive that $\operatorname{Det}(A) = \operatorname{Det}(B)$. Here, one may consider the following cases:

**Case 1.** $A$ is invertible. This implies that $B$ is invertible. It follows from $\operatorname{Det}(AB^{-1}) = 1$ that $A = CB$ for some $C \in SL_n(D)$. Since $C$ is a product of commutators, we get $C \widehat{\sim}_1 I_n$. This immediately gives that $A \widehat{\sim}_1 B$.

**Case 2.** $A$ is singular. Then $A$ is a product of idempotent matrices. Any idempotent matrix $E \in M_n(D)$ is similar to any diagonal matrix having only 0 and 1 on the diagonal with the number of 1's is equal to the rank of $E$. So, for any $1 \leq i \leq n$, there is an invertible matrix $P_i$ such that $P_i E P_i^{-1}$ is a diagonal matrix with a zero on the $(i, i)$ entry, and we thus have $P_1 E P_1^{-1} P_2 E P_2^{-1} \cdots P_n E P_n^{-1} = 0$. We can thus write

$$E = E^n \widehat{\sim}_1 P_1 E P_1^{-1} P_2 E P_2^{-1} \cdots P_n E P_n^{-1} = 0.$$

Since $A$ is a product of idempotent matrices, we get that $A\widehat{\sim}_1 0$ and similarly for the singular matrix $B$. This yields the proof. □

Proposition 4.6 and Theorem 4.7 give the following theorem.

**Theorem 4.8.** *Let $R$ be an Artinian semisimple ring. Then* $\operatorname{diam}_\wedge(R) \leq 2$.

In the case of a matrix ring over a field we have a better bound:

**Theorem 4.9.** *Let $F$ be a field and $n \in \mathbb{N}$. Then* $\operatorname{diam}_\wedge(M_n(F)) = 1$.

*Proof.* We have seen in the proof of Theorem 4.7 that in the case when two matrices $A, B$ are invertible, we have $A\widehat{\sim}_1 B$. So let us consider the case when two matrices $A$ and $B$ are singular. Without loss of generality, we may assume that $\operatorname{rank}(A) \leq \operatorname{rank}(B)$. In this case, the matrix $A$ can be written as products of conjugates of $B$ (cf. [3]), and we obtain that there exist $k \in \mathbb{N}$ and $P_1, \ldots, P_k \in GL_n(F)$ such that $A = P_1 B P_1^{-1} \cdots P_k B P_k^{-1}$. This immediately gives that $A\widehat{\sim}_1 B$, as desired. □

Turning our attention to the $\operatorname{diam}_*$, we can express the distance $d_*$ between two invertible matrices as follows.

**Proposition 4.10.** *Let $n \in \mathbb{N}$, and let $D$ be a division ring such that $n \neq 2$ and $D \neq \mathbb{F}_2$. Let $A, B \in GL_n(D)$ be two matrices such that $B \in \widehat{\{A\}}$. Then $AB^{-1} \in SL_n(D)$ and $d_*(A, B)$ is the minimal number of commutators required to express $AB^{-1}$ as products of commutators in $GL_n(D)$.*

*Proof.* We have seen in Theorem 3.20 that $B \in \widehat{\{A\}}$ implies $AB^{-1} \in GL_n(D)'$, the derived subgroup of $GL_n(D)$. Our assumption implies that $GL_n(D)' = SL_n(D)$ (cf. [6, Theorem 4 on page 138]). Assume that $d_*(A, B) = \ell$ and that we have a chain $A = A_0 \overset{*}{\sim}_1 A_1 \overset{*}{\sim}_1 \ldots \overset{*}{\sim}_1 A_\ell = B$ We thus get, for any $0 \leq i \leq \ell - 1$, invertible matrices $X_i, Y_i, Z_i \in GL_n(D)$ such that $A = A_0 = X_0 Y_0 Z_0, A_1 = X_0 Z_0 Y_0 = X_1 Y_1 Z_1, A_2 = X_1 Z_1 Y_1 = X_2 Y_2 Z_2, \ldots, A_\ell = X_{\ell-1} Z_{\ell-1} Y_{\ell-1} = B$. From this, we deduce that $B = A_\ell = A_{\ell-1}[Z_{\ell-1}^{-1}, Y_{\ell-1}^{-1}]$ and $A_{\ell-1} = A_{\ell-2}[Z_{\ell-2}^{-1}, Y_{\ell-2}^{-1}]$. Continuing this process, we conclude the following equality

$$B = A[Z_0^{-1}, Y_0^{-1}][Z_1^{-1}, Y_1^{-1}] \cdots [Z_{\ell-1}^{-1}, Y_{\ell-1}^{-1}].$$

Conversely, if such an equality holds, we can deduce immediately that $d_*(A, B) \leq \ell$. This finishes the proof. □

On account of Proposition 4.10, we find that $\operatorname{diam}_*(SL_n(D))$ is strongly related to the minimal number of commutators needed to express an element of the derived group of the division ring $D$. The reader interested by this topics can consult the recent paper Gvozdevsky [7] and the bibliography mentioned there.

The other class we need to look at for computing the $*$-diameter in matrix rings over divison rings is the class of singular matrices. This is the purpose of the following proposition. We have already use the fact that any singular matrix with entries in a division ring $D$ can be presented as a product of idempotent matrices. In fact, we can be a bit more precise: any singular matrix $A \in M_n(D)$ is a product of conjugates of the idempotent matrix $E = \operatorname{diag}(1, \ldots, 1, 0)$ (cf. [8]). This will be used in the proof of the statement (ii) of the following proposition.

**Proposition 4.11.** *(i) Let $F^2 = F \in M_n(D)$ be of rank $k$. Then* $d_*(F, 0) \leq \lceil n/(n-k) \rceil$.

*(ii) Let $A, B \in M_n(D)$ be two singular matrices, and let $A$ (respectively, $B$) be a product of $k \geq 1$ (respectively, $l \geq 1$) matrices similar to $E = \mathrm{diag}(1, \ldots, 1, 0)$. Then $d_*(A, B) \leq k + l$.*

*(iii)* $\mathrm{diam}_*(\widehat{\{0\}}) \leq 2n$.

*Proof.* (i) $F$ is similar to any diagonal matrix having $k$ elements 1 and $n-k$ elements 0 on the diagonal. In order to cover the $n$ positions on the diagonal with strips of zeros of length $n-k$, we need at most $l := \lceil n/(n-k) \rceil$ strips. For $i = 1, \ldots, l$, let $P_i \in GL_n(D)$ be such that $P_i F P_i^{-1}$ is a diagonal matrix with $n-k$ zeros on the diagonal occupying the positions from $(i-1)(n-k)+1$ till $\min\{i(n-k), n\}$ positions along the diagonal (and 1's in the remaining places). We thus get

$$\begin{aligned} F = F^l &\overset{*}{\sim}_1 P_1 F P_1^{-1} F^{l-1} \\ &\overset{*}{\sim}_1 P_1 F P_1^{-1} P_2 F P_2^{-1} F^{l-2} \\ &\overset{*}{\sim}_1 P_1 F P_1^{-1} P_2 F P_2^{-1} P_3 F P_3^{-1} F^{l-3} \\ &\overset{*}{\sim}_1 \\ &\vdots \\ &\overset{*}{\sim}_1 P_1 F P_1^{-1} \cdots P_l F P_l^{-1} = 0, \end{aligned}$$

Where the last equality is due to the fact that all the positions along the diagonal appear to have a zero in at least one of the factors $P_i F P_i^{-1}$. From this, we indeed obtain that $d_*(F, 0) \leq l$.

(ii) By hypothesis, we know that there exist invertible matrices $P_1, \ldots, P_k \in GL_n(D)$ and $Q_1, \ldots, Q_l \in GL_n(D)$ such that

$$A = P_1 E P_1^{-1} P_2 E P_2^{-1} \cdots P_k E P_k^{-1},$$

and

$$B = Q_1 E Q_1^{-1} Q_2 E Q_2^{-1} \cdots Q_l E Q_l^{-1}.$$

This immediately gives the following paths between $A$, $B$, and $E$:

$$\begin{aligned} A &\overset{*}{\sim}_1 P_2 E P_2^{-1} P_3 \cdots P_k E P_k^{-1} E \\ &\overset{*}{\sim}_1 P_3 E P_3^{-1} P_4 \cdots P_k E P_k^{-1} E^2 \\ &\overset{*}{\sim}_1 \\ &\vdots \\ &\overset{*}{\sim}_1 P_k E P_k^{-1} E^{k-1} \sim_1 E^k = E. \end{aligned} \tag{1}$$

A similar path is linking $B$ and $E$. This leads to $d_*(A, B) \leq k + l$.

(iii) Assume that $A \in M_n(D)$ is singular. Then there exist invertible matrices $P_1, \ldots, P_k \in GL_n(D)$ such that $A = P_1 N P_1^{-1} P_2 N P_2^{-1} \cdots P_k N P_k^{-1}$, where $N := \sum_{i=1}^{n-1} E_{i,i+1}$. Now, one may consider the following cases:

**Case 1.** $k < n$. Then, as in the proof of part (ii) above (cf. equations (1)), we have $d_*(A, N^k) \leq k$. Since the nilpotent index of $N^k$ is $n-k$, Theorem 3.7 in [1] shows that $d_c(N^k, 0) \leq n-k-1$, and by virtue of $d_*(N^k, 0) \leq d_c(N^k, 0)$, we gain $d_*(N^k, 0) \leq n-k-1$. This gives rise to

$$d_*(A, 0) \leq d_*(A, N^k) + d_*(N^k, 0) = k + (n-k-1) = n-1.$$

**Case 2.** $k \geq n$. Then, as in the proof of part (ii) above, we have

$$A \overset{*}{\sim}_n P_{n+1} N P_{n+1}^{-1} P_{n+2} \cdots P_k^{-1} N^n = 0.$$

This shows that $d_*(A, 0) \leq n$.

If $A, B \in \widehat{\{0\}}$, we then have $d_*(A, B) \leq d_*(A, 0) + d_*(0, B) \leq 2n$, as claimed. □

We end the paper with some considerations on the structure of upper triangular matrix rings. Recall that a *strictly upper triangular* matrix is an upper triangular matrix having 1's along the diagonal and 0's under it, i.e., a matrix $A = [a_{i,j}]$ such that $a_{i,j} = 0$ for all $i \geq j$ and $a_{ii} = 1$. We denote the set of all $n \times n$ strictly upper triangular matrix over a ring $R$ by $U_n(R)$.

Let us recall that, for a ring $R$ and $n \in \mathbb{N}$, we denote $N_n(R)$ as the set of elements of $R$ that are nilpotent of index $n$ and that a ring $R$ is called semi-commutative if, for $a, b \in R$, whenever $ab = 0$ we have $aRb = 0$.

**Theorem 4.12.** *Let $R$ be a ring. Then the following statements hold:*

*(i) If $R$ is semi-commutative, then, for each $i \in \mathbb{N}$, we have $\{0\}_i^* \subseteq N_{2^i}(R)$. In particular, $\{0\}^* \subseteq N(R)$.*

*(ii) For any strictly upper triangular matrix $U \in M_n(R)$, we have*
  - (a) $U \in \widehat{\{0\}}_{n-1} \subseteq \widehat{\{0\}}$ *and* $U \in \{0\}_{n-1}^* \subseteq \{0\}^*$.
  - (b) $\operatorname{diam}_\wedge(U_n(R)) \leq 2(n-1)$ *and* $\operatorname{diam}_*(U_n(R)) \leq 2(n-1)$ *for all* $n \geq 2$.

*Proof.* (i) We argue by induction on $i$. Let $i = 1$. We show that $\{0\}_1^* \subseteq N_2(R)$. Pick an arbitrary element $z \in \{0\}_1^*$. This implies that $z \overset{*}{\sim}_1 0$, and hence there exist elements $a_1, a_2, a_3 \in R$ such that $z = a_1a_2a_3$ and $0 = a_1a_3a_2$. Since $R$ is semi-commutative, one obtains $a_1Ra_3Ra_2R = (0)$. This leads to $z^2 = 0$, that is, $z \in N_2(R)$. hence the claim is true for the case $i = 1$. Now, suppose, inductively, that $i > 0$ and that the result has been shown for all values less than $i + 1$, in particular, $\{0\}_i^* \subseteq N_{2^i}(R)$. Let $z \in \{0\}_{i+1}^*$. This means that $z \overset{*}{\sim}_1 y \overset{*}{\sim}_i 0$ for some $y \in R$. Since $y \in \{0\}_i^*$, the induction hypothesis gives that $y^{2^i} = 0$. In addition, it follows from $z \overset{1}{\sim} y$ that there exist elements $a_1, a_2, a_3 \in R$ such that $z = a_1a_2a_3$ and $y = a_1a_3a_2$. On account of $R$ is semi-commutative, one can concule from $y^{2^i} = 0$ that $(a_1Ra_3Ra_2R)^{2^i} = (0)$. This gives rise to $z^{2^{i+1}} = 0$, and thus $z \in N_{2^{i+1}}(R)$. This completes the inductive step, and so the claim has been proven by induction.

To establish the last assertion, it is enough to observe that $\{0\}^* = \bigcup_{i \geq 0} \{0\}_i^*$ and $N(R) = \bigcup_{n \geq 1} N_n(R)$.

(ii) We first show (a). According to [1, Proposition 3.1], for any strictly upper triangular matrix $U \in M_n(R)$, we have $U \in \{0\}_{n-1} \subseteq \overline{\{0\}}$. Now, the claim can be deduced based on the facts $\{0\}_{n-1} \subseteq \{0\}_{n-1}^*$ and $\{0\}_{n-1} \subseteq \widehat{\{0\}}_{n-1}$.

To demonstrate (b), fix $n \geq 2$, and let $A$ and $B$ be two arbitrary elements in $U_n(R)$. It follows from part (a) that $U \in \{0\}_{n-1}^* \subseteq \{0\}^*$ (respectively, $U \in \widehat{\{0\}}_{n-1} \subseteq \widehat{\{0\}}$). Thus, one can conclude that $A \overset{n-1}{\sim} 0 \overset{n-1}{\sim} B$ (respectively, $A \widehat{\sim}_{n-1} 0 \widehat{\sim}_{n-1} B$). We therefore get

$$\operatorname{diam}_*(U_n(R)) \leq 2(n-1) \quad (\text{respectively, } \operatorname{diam}_\wedge(U_n(R)) \leq 2(n-1)).$$

This completes the proof. □

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